the genertalized Numerical Range and its Boundary

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the genertalized Numerical Range and its Boundary

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Abstract. *We consider the boundary of $C-$numerical range of a matrix and its related topics in the theory of invariant polynomials*

1. the generalized numerical range of a matrix as a semi-algebraic set.

We recall the definition of the $C-$numerical range. Denote by $M_n(\mathbf{C})$ the set of all $n \times n$ complex matrices.

Def. *Let A, C be $n \times n$ complex matrices. Set*

$$W_C(A) = \{tr(CUAU^*) : U \in U(n)\} \tag{1.1}$$

where $U(n)$ is the compact group of all unitary matrices of oder n. $W_C(A)$ is called the C-numerical range of A.

It is clear that C-numerical range is a unitary invariant of the suare matrix A: If B is unitarily similar to A ,i.e, $B = U\ A\ U^*$, then $W_C(B) = W_C(A)$. If C is a rank one orthogonal projection, then the C-numerical range $W_C(A)$ coincides with the classical numerical range $W(A)$,

$$W(A) = \{(A\xi, \xi) : \xi \in \mathbf{C}^n, ||\xi|| = 1\}.$$

The generalized numerical range $W_C(A)$ is the range of the real algebraic variety

$$U(n) = \{U = \{u_{i,j}\}_{1 \leq i,j \leq n} \in \mathbf{C}^{n^2} \cong \mathbf{R}^{2n^2} : \sum_{k=1}^{n} u_{i,k}\overline{u_{j,k}} = \delta_{i,j}\ (1 \leq i,j \leq n)\} \tag{1.2}$$

under the polynomial mapping

(1.3). $\quad X\ (\in M_n(\mathbf{C}) = \mathbf{C}^{n^2} \cong \mathbf{R}^{2n^2}) \mapsto\ tr(CXAX^*)\ (\in \mathbf{C} \cong \mathbf{R}^2)$

We recall Tarski=Seidenberg's theorem.

Def. *A subset Λ of the $n-$dimensional affine space $\mathbf{R}^n$ is said to be semi-algebraic if Λ is an element of the Boolean algebra generated by the family of sets*

$$Z_0(f) = \{x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : f(x_1, x_2, \ldots, x_n) = 0\},$$

and

$$Z_1(f) = \{x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : f(x_1, x_2, \ldots, x_n) > 0\}$$

where f varies over the real polynomial ring $\mathbf{R}[X_1, X_2, \ldots, X_n]$.

In other words, Λ :a semi-algebraic set $(\subset \mathbf{R}^n)$ $\Leftrightarrow_{\textbf{definition}}$

$$\Lambda = \cup_{j=1}^{m} \cap_{k=1}^{l_j}\ Z_{\epsilon_{j,k}}(f_{j,k})$$

where $\epsilon_{j,k} \in \{0,1\}, f_{j,k} \in \mathbf{R}[X_1, X_2, \ldots, X_n]$.

Of course, an algebraic set $(\subset \mathbf{R}^n)$ is semi-algebraic .

a Corollary of Tarski=Seidenberg's theorem.

The range $\phi(\Lambda)$ of a semi-algebraic set $\Lambda \subset \mathbf{R}^m$ under a polynomial mapping $\phi = (\phi_1, \phi_2, \ldots, \phi_n) : \mathbf{R}^m \to \mathbf{R}^n$, is semi-algebraic, where $\phi_j \in \mathbf{R}[X_1, X_2, \ldots, X_m]$ $(1 \leq j \leq n)$.

A Simple Example

$$\mathbf{S}^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbf{R}^3$$

$\downarrow$ projection

A closed disc

$$\{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 \leq 1\}.$$

We remark that the closed disc is semi-algebraic and not algebraic.

Another Result about Semi-Algebraic Sets. *The boundary $\partial\Lambda$ of a semi-algebraic set $\Lambda(\subset \mathbf{R}^n)$ is also semi-algebraic.*

Using the result above, Takaguchi,Nishikawa and I proved the following.

Theorem. *[Takaguchi=Nishikawa=Nakazato[1]] The bounbary $\partial W_C(A)$ of $C-$numerical range $W_C(A) \subset \mathbf{C} \cong \mathbf{R}^2$ is the union of a finite number of real algebraic arcs.*

Furthermore, using the theorem above and Bezout's theorem, we proved the following.

Theorem. *The number of non $\omega-$regular points of $\partial W_C(A)$ is finite.*

2. Concrete C-numerical ranges

Second I consider the $q-$numerical ranges of matrices.

We set

$$C = C_q = \begin{pmatrix} q & \sqrt{1-|q|^2} & 0 & 0\dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

We denote $W_{C_q}(A)$ by $W(A:q)$. We recall N.K.Tsing's formula.

A formula due to Tsing.

$$W(A:q) = \cup\{q(A\xi,\xi) + \sqrt{1-|q|^2}r\exp(i\theta)\sqrt{||A\xi||^2 - |(A\xi,\xi)|^2} : \xi \in \mathbf{C}^n, ||\xi|| = \mathbf{1}, \mathbf{0} \le \mathbf{r} \le \mathbf{1}, \mathbf{0} \le \theta \le \mathbf{2}\pi\} \quad (2.2).$$

This formula is rewriten as

$$W(A:q) = \cup\{B(qz : \sqrt{\phi_A(z)} : z \in W(A:1)\} \qquad (2.3),$$

where $\phi_A(z) = \max\{||A\xi||^2 - |z|^2 : \xi \in \mathbf{C}^n, ||\xi|| = \mathbf{1}, (\mathbf{A}\xi,\xi) = \mathbf{z}\}$ **(2.5)**,
for $z \in W(A) = W(A:1)$.
We set

$$\psi_A(z) = \max\{||A\xi||^2 : \xi \in \mathbf{C}^n, ||\xi|| = \mathbf{1}, (\mathbf{A}\xi,\xi) = \mathbf{z}\}.$$

We review N.K.Tsin's results.

Theorem. [N.K.Tsing [6]] ψ_A is a concave function on $W(A)$ and hence ϕ_A and $\sqrt{\phi_A}$ are also concave on $W(A)$.

Using the theorem above we can prove the following two claims.

Proposition. $A, B_1, \dots, B_m$ *: matrices.* $W(A) = \cup_{j=1}^m W(B_j)$, $\phi_A(z) = \max\{\phi_{B_j}(z) : 1 \le j \le m\}$,

$$\Rightarrow W(A:q) = \cup_{j=1}^m W(B_j:q)$$

for every q with $|q| \le 1$.

Theorem. [2]] (1) Suppose that $\lambda_1, \lambda_2, \dots, \lambda_n$ are arbitrary complex numbers ($n \ge 3$). Then the following equation holds for every $z \in \mathrm{Conv}(\{\lambda_1, \lambda_2, \dots, \lambda_n\})$:

$$\phi_{\mathrm{diag}(\lambda_1,\lambda_2,\dots,\lambda_n)}(z) = \max\{\phi_{\mathrm{diag}(\lambda_{j_1},\lambda_{j_2},\lambda_{j_3})}(z) : 1 \le j_1 < j_2 < j_3 \le n\}.$$

Hence

$$W(\mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) : q) = \cup\{\mathrm{W}(\mathrm{diag}(\lambda_{j_1}, \lambda_{j_2}, \lambda_{j_3}) : q) : 1 \le j_1 < j_2 < j_3 \le n\}.$$

(2) If $\lambda_1, \lambda_2, \lambda_3 \in \mathbf{C}$ and $|\lambda_1| = |\lambda_2| = |\lambda_3| = 1$,then

$$\partial W(\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) : q) \subseteq \{z \in \mathbf{C} : |z| = 1\} \cup \{\partial \mathrm{W}(\mathrm{diag}(\lambda_j, \lambda_k) : q) : 1 \le j < k \le 3\}.$$

3. C-numerical ranges of 2 by 2 matrices
The study of $C-$numerical ranges of 2×2 matrices is reduces to that of

$$W(\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} : q)$$

for $0 \leq \beta \leq \alpha, 0 \leq q \leq 1$.

We have the following formula(cf.[3]):

$$W(\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} : q)$$

$=\{r[(\alpha+\beta)/2+\sqrt{1-q^2}(\alpha-\beta)/2]\cos(\theta)+\sqrt{-1}r[(\alpha-\beta)/2+\sqrt{1-q^2}(\alpha+\beta)/2]\sin(\theta) : 0 \leq r \leq 1,$
$0 \leq \theta \leq 2\pi\}$, *where* $0 \leq \beta \leq \alpha, 0 \leq q \leq 1$.

4. Relation between generalized numerical ranges and invariant polynomials
It is well known that the following three conditions for 2×2 matrices x, y are mutually equivalent: (i) $W(x) = W(y)$;(ii) $\mathrm{tr}(x)=\mathrm{tr}(y), \mathrm{tr}(x^2) = tr(y^2), tr(x^*x) = tr(y^*y)$; (iii) $y = UxU^*$ for some unitary matrix.

For $n \times n$ matrices x, y the following three conditions are mutually equivalent: (iv) $W_P(x) = W_P(y)$ for every orthogonal projection P ;(v) $W_H(x) = W_H(y)$ for every Hermitian matrix H ; (vi) $\det(\lambda\, I_n - (\alpha x + \beta x^*)) = \det(\lambda\, I_n - (\alpha y + \beta y^*))$ for every $\lambda, \alpha, \beta \in \mathbf{C}$.

In the case $n = 3$, the condition (iv) is equivalent to the condition: $W(x) = W(y), \mathrm{tr}(x) = \mathrm{tr}(y)$.

Question. x, y: 3×3 *matrices such that*

$$\mathrm{tr(x) = tr(y), W(x : q) = W(y : q)}$$

for every $0 \leq q \leq 1$.

$$\Rightarrow_? \ {}^\exists U : \text{a unitary matrix such that } \mathrm{y = UxU^* or\ y = U({}^tx)U^*}.$$

I had the question above under the influence of the following theorem.

Theorem. (S.Teranishi,Nagoya Math.J.1986[8]) . *Let x, y be arbitrary 3×3 complex matrices. (1) If*

$$\mathrm{tr(x^k) = tr(y^k)\ (k = 1, 2, 3)} \tag{4.1}$$

$$\mathrm{tr(x^*x) = tr(y^*y)} \tag{4.2}$$

$$\mathrm{tr(x^{*2}x) = tr(y^{*2}y)} \tag{4.3}$$

$$\mathrm{tr(x^{*2}x^2) = tr(y^{*2}y^2)} \tag{4.4}$$

then there exits a unitary matrix U such that

$$y = UxU^* \text{ or } \mathrm{y = U({}^tx)U^*}$$

. *(2) If the conditions (4.1),(4.2),(4.3),(4.4) holds and if*

$$\mathrm{tr(x^{*2}x^2x^*x) = tr(y^{*2}y^2y^*y)} \tag{4.5}$$

then there exits a unitary matrix U such that $y = UxU^*$.

We denote by $M_n(\mathbf{C})$ the C^*-algebra of all $n \times n$ complex matrices. I want to find a *finite* subset F of $M_n(\mathbf{C})$ such that $\{W_C(\cdot) : C \in F\}$ is a complete unitary invariant system(cf.[7],Theorem 10). This desire is influenced by the following theorem.

Theorem (cf.[9]) Let x, y be $n \times n$ complex matrices. *Set* $x_1 = x, x_2 = x^*, y_1 = y, y_2 = y^*$. *If*

$$\mathrm{tr}(\mathrm{x_{i_1}\ x_{i_2}\ \cdots x_{i_p}}) = \mathrm{tr}(\mathrm{y_{i_1}\ y_{i_2}\ \cdots x_{i_p}}).$$

for every $1 \leq p \leq 2^n - 1, 1 \leq i_1, i_2, \ldots, i_p \leq 2$, *then there exits a unitary matrix* U *such that* $y = UxU^*$.

We can obtain another criterion by using the following lemma.

Lemma. *Let* A, B *be* C^**−subalgebras of* $M_n(\mathrm{C})$ *such that* $I_n \in A, I_n \in B$. *If* ϕ *is a* $*$*−isomorphism of* A *onto* B *satisfying* $\mathrm{tr}(\phi(\mathrm{x})) = \mathrm{tr}(\mathrm{x})$ *for every* $x \in A$, *then there exits a unitary matrix* U *such that* $\phi(x) = UxU^*$ *for every* $x \in A$.

Proposition. *If*

$$\mathrm{tr}(\mathrm{x_{i_1}\ x_{i_2}\ \cdots x_{i_p}}) = \mathrm{tr}(\mathrm{y_{i_1}\ y_{i_2}\ \cdots y_{i_p}})$$

for every $1 \leq p \leq 4\,(n^2-3)$, $1 \leq i_1, i_2, \ldots, i_p \leq 2$, *then there exists a unitary matrix* U, *such that* $y = UxU^*$.

This proposition is stronger than the theorem above under the condition

$$n \geq 8.$$

We have an analogous reslut.

Proposition. *If*

$$\mathrm{tr}(\mathrm{x_{i_1}\ x_{i_2}\ \cdots x_{i_q}}) = \mathrm{tr}(\mathrm{y_{i_1}\ y_{i_2}\ \cdots y_{i_q}})$$

for every $1 \leq q \leq 2\,(n^2 - 3)$, $1 \leq i_1, i_2, \ldots, i_q \leq 2$, *then there exists a *-preserving, Hilbert-Schmidt norm preserving linear bijection of the* C^**-algebra* $C^*(x)$ *onto the* C^**-algebra* $C^*(y)$.

5. 3 by 3 orthostochastic matrices

We consider a 3×3 real matrix

$$A(x,y,z:u,v,w) = \begin{pmatrix} x+w & y+u & z+v \\ z+u & x+v & y+w \\ y+v & z+w & x+u \end{pmatrix} = xe_1 + ye_2 + ze_3 + ue_4 + ve_5 + we_6,$$

for $x \geq 0, y \geq 0, z \geq 0, u \geq 0, v \geq 0, w \geq 0, x+y+z+u+v+w = 1$. Here

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$e_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Such a matrix $A(x,y,z:u,v,w)$ is called a 3×3 doubly stochastic matrix. A (3×3) doublystochastic matrix D is said to be orthostochastic if it is the Hadamard product (entrywise product) of some (3×3) unitary matix U and its conjugate matrix $\overline{U}$. The necessary and sufficient condition of Au-Yeng and Poon for a doubly stochastic matix

$$A(x,y,z:u,v,w)$$

to be orthostochastic (cf.[5]) is stated in anther fashion(cf.[4]):

$$\begin{aligned} &x^2y^2 + x^2z^2 + y^2z^2 - 2xyz(x+y+z) + u^2v^2 + u^2w^2 + v^2w^2 - 2uvw(u+v+w) \\ &-4uvw(x+y+z) - 4xyz(u+v+w) - 2(xy+xz+yz)(uv+uw+vw) \leq 0 \quad . \end{aligned} \tag{5.1}$$

Using this characterization we can obtain the following proposition.

Proposition ([4]) Suppose that A_3 is the affine hull of the compact convex set of all 3×3 doubly stochastic matrices and O_3 is the compact set of all 3×3 orthostochastic matrices. *Denote by ∂O_3 the boundary of O_3 in the affine space A_3. Then the following equation holds:*

$$\partial O_3 = \{g \circ g : g \in SO(3)\},$$

where $SO(3)$ is the group of rotations in the 3-dimensional Euclidean space.

Consider a linear functional Ψ of the complex vector space $M_3(\mathbf{C})$,

$$\Psi : X = \{x_{i,j} : 1 \leq i,j \leq 3\} \in M_3(\mathbf{C}) \mapsto \ (x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33}).$$

Suppose that A and C are complex 3×3 diagonal matrices with diagonal entriries

$$a = \{a_1, a_2, a_3\}, c = \{c_1, c_2, c_3\}$$

respectively. Consider the 3×3 matrix $[A : C]$ given as the product $({}^t a)\ c$,that is,

$$[A : C] = \begin{pmatrix} a_1c_1 & a_1c_2 & a_1c_3 \\ a_2c_1 & a_2c_2 & a_2c_3 \\ a_3c_1 & a_3c_2 & a_3c_3 \end{pmatrix}.$$

Then the $C-$ numerical range of A coincides with the set

$$\{\Psi([A : C] \circ G) : G \in O_3\}.$$

The following proposition is effective to determine the $C-$ numerical range of a matrix A under the condition $A = C$ and A is a 3×3 complex diagonal matrix.

Proposition ([4]) Suppose that A ia a complex 3×3 complex diagonal matrix. *Then the A-numerical range $W_A(A)$ coincides with the set*

$$\{\Psi([A : A] \circ G) : G \in O_3 \text{ and } ({}^t\ \mathrm{G}) = \mathrm{G}\}.$$

Under the condition

$$A = \mathrm{diag}\{1, 0, \alpha\},$$

the description of the boundary of the range $W_A(A)$ concerns deeply the curve " (Steiner's) Deltoid". We define the Deltoid $D[z_0 : K : \exp[i\ \eta]]$ (z_0 is a complex number,K is a positive number, η is a real number) in the complex plane C by the following : Choose $\exp[i\ \theta]$ (θ is a real number) ia an arbitratry cubic root of $\exp[i\ \eta]$.

$$D[z_0 : K : \exp[i\ \eta]] = \{z_0 + (K/3)\ \ \exp[i\ \theta](2\exp[2\ i\phi] + \exp[-i\phi]) : 0 \leq \phi \leq 2\ \pi\}.$$

We obtain the following theorem.

Theorem ([4]) Suppose that $A = \mathrm{diag}\{1, 0, \alpha\}$ and $\alpha = a + i\ b$ where a, b are real numbers with $b \neq 0$. *Then the boundary of the range $W_A(A)$ is a subset of the following set*

$$\{(1-t)(1+\alpha^2) + t : 0 \leq t \leq 1\} \cup \{(1-t)(1+\alpha^2) + 2t\alpha : 0 \leq t \leq 1\} \cup \{(1-t)(1+\alpha^2) + t(\alpha)^2 : 0 \leq t \leq 1\}$$

$$\cup D[z_0 : K : \exp[i\ \eta]]$$

where

$$z_0 = (1/(4\ b^2))\alpha(\overline{\alpha} - 1)(3a^2 - 3a - b^2 + i\ [b + 4ab]),$$

$$K = (3/(4\ b^2))|\alpha - 1|^2\ |\alpha|^2,$$

$$|\alpha|^2\ |\alpha-1|^2 \exp[i\ \eta] = -(a-a^2+b-2a\ b+b^2)(a-a^2-b+2a\ b+b^2)+i(-2\ b)(2a-1)(a^2-a-b^2).$$

We give you some graphics about $C-$numerical ranges after this document.

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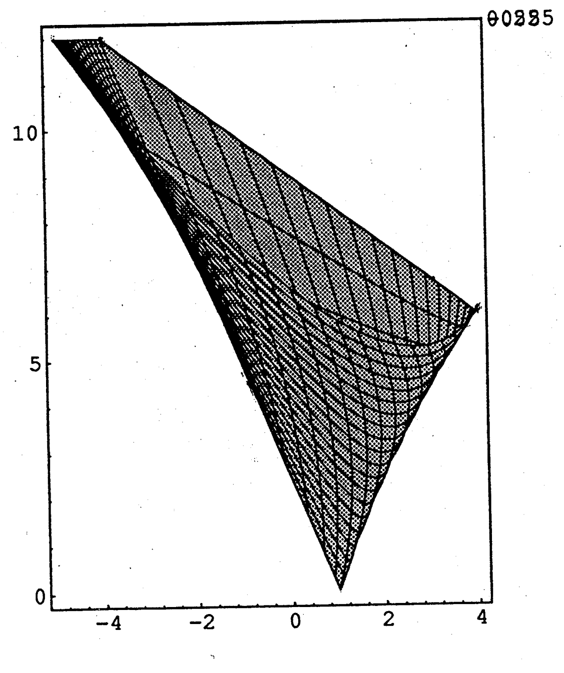
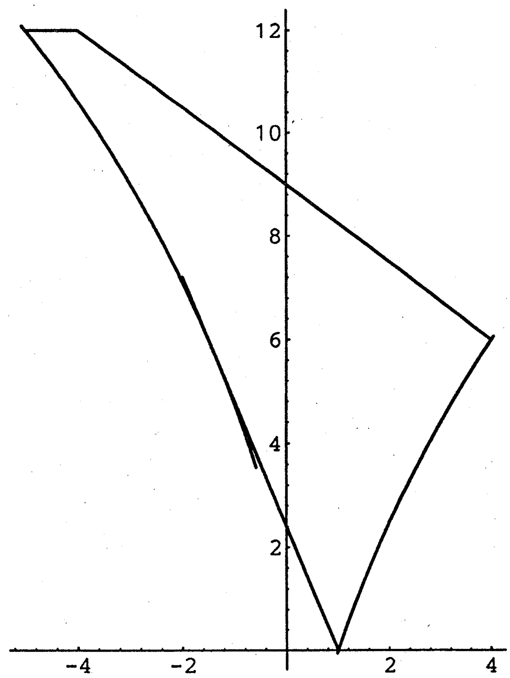




$W_C(C)$

(II-3) C=diag{ 1 , 0, 2 +3 i }.

Graphics of C-numerical Ranges of Matrices and the set of 3 x 3 symmetric orthostochastic matrices

(I)The q-numerical range of a 3 x 3 complex diagonal matrix A .

(i) The diagonal entries of A are mutually distinct cubic roots of 1.

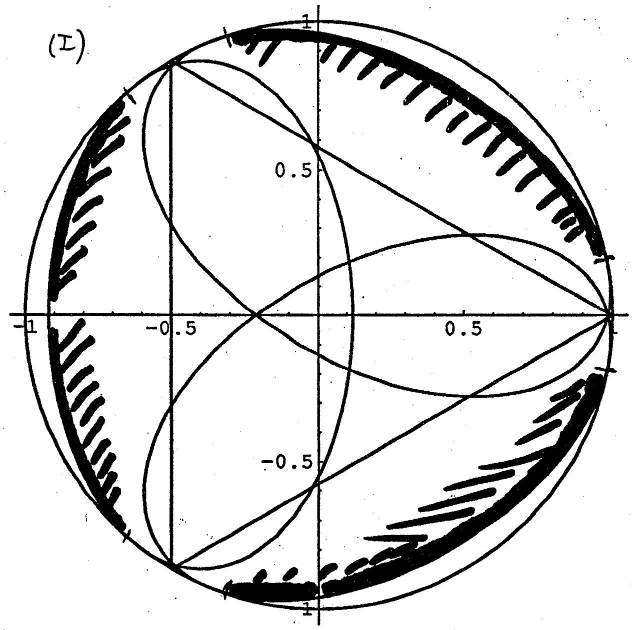
(ii) q=4/5 .

(II)The C-numerical range $W_c(C)$ of the 3 x 3 complex diagonal matrix.

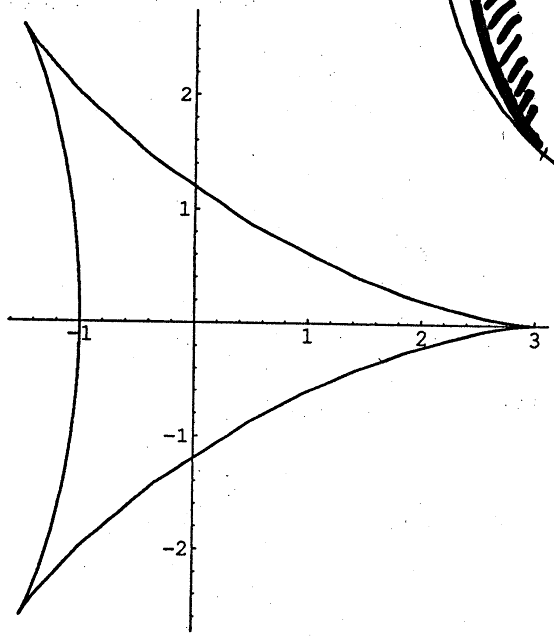
(II-1) The diagonal entries of C are mutually disdistinct cubic roots of 1. This generalized numerical range coincides with the closed domain surounded by the Deltoid D(0,3,1):

x=2 cos(2 t) +cos(2t),y=2sin(2t) -sin(2t)

(I)

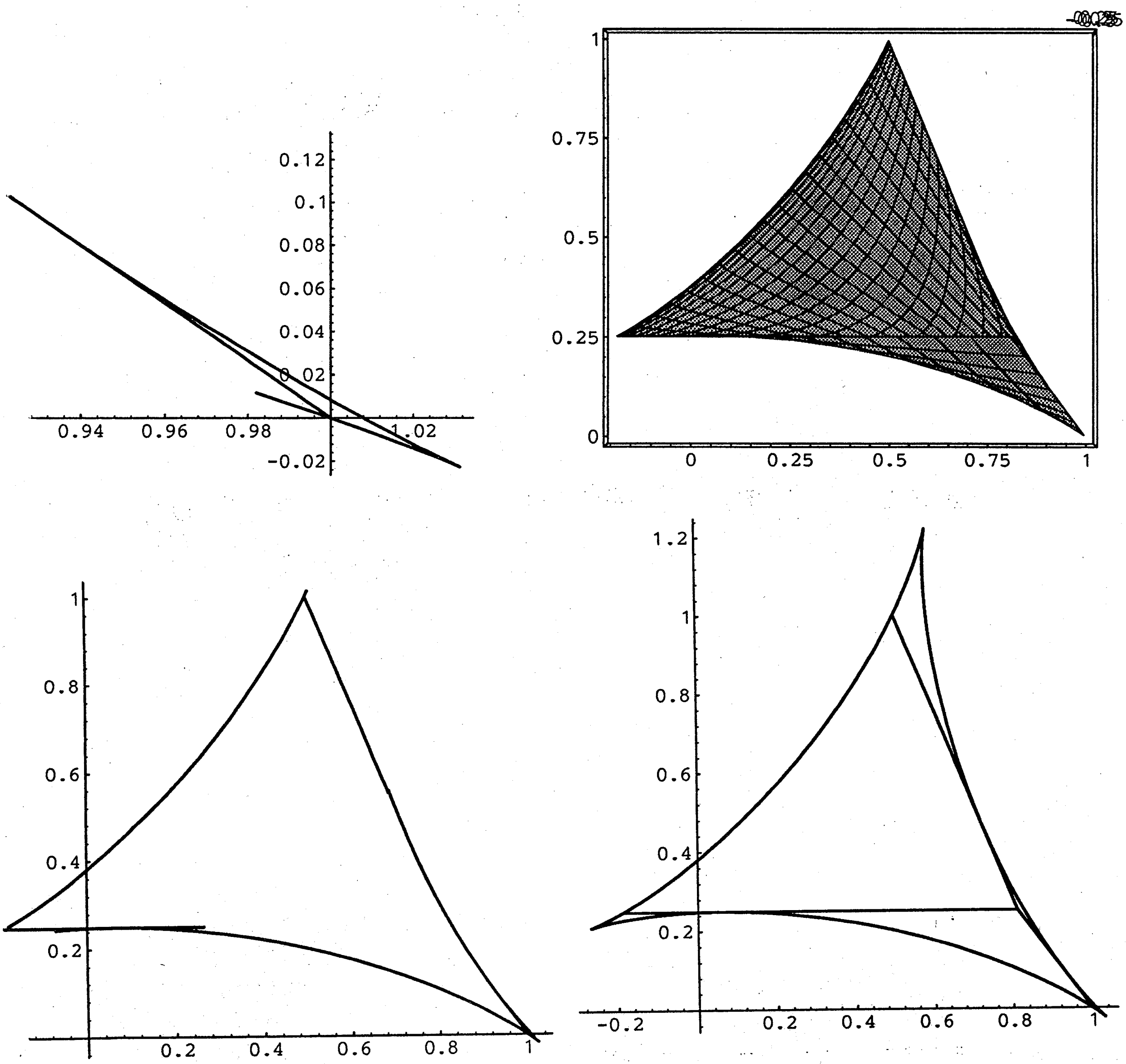


(II-1)



(II-2) $C = \mathrm{diag}\{ 1 , 0, (1/4) + i\,(1/2)\}$.

One graphic describes the range $W_c(C)$ based on the command " ParametricPlot3D" (2-parameter plot) in " Mathematica" (ver2.0, hardware : Apple,Quadra 900) . Other three graphics describe its boundary based on the command " ParametricPlot" (1-parameter plot on the plane).



(III) The set of 3 x 3 symmetric orthostochastic matrices in the 3-dimensional affine space H of all 3 x 3 symmetric real matrices $\{(a_{11},a_{12},a_{13}),(a_{12},a_{22},a_{23}),(a_{13},a_{23},a_{33})\}$ such that

$a_{11}+a_{12}+a_{13}=a_{12}+a_{22}+a_{23}=a_{13}+a_{23}+a_{33}=1$.

We use a non-orthogonal coordinates system < , ,> in H for which:

$\{(1/3,1/3,1/3),(1/3,1/3,1/3),(1/3,1/3,1/3)\}=\langle 0,0,0\rangle$,

$\{(0,1,0),(1,0,0),(0,0,1)\}=\langle 1,0,0\rangle$,

$\{(0,0,1),(0,1,0),(1,0,0)\}=\langle -1/2,\sqrt{3}/2,0\rangle$,

$\{(1,0,0),(0,0,1),(0,1,0)\}=\langle -1/2,-\sqrt{3}/2,0\rangle$,

$\{(1,0,0),(0,1,0),(0,0,1)\}=\langle 0,0,b\rangle$

for some $b > 0$.